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# Ampleness Criteria for Line Bundles on Algebraic Stacks

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## Abstract

We define **"ampleness"** for line bundles on algebraic stacks. There exist certain equivalent criteria for line bundles on Deligne-Mumford stacks. For Artin stacks, these criteria are no longer equivalent, but we can give a definition of ampleness using a variant of Seshadri constant.

## Introduction

$X$ : scheme / $\mathbb{C}$, $L$ : line bundle on $X$.

- $L$ is **very ample** $\iff$ $\varphi_L : X \to \mathbb{P}^n$ is a closed immersion.
- $L$ is **ample** $\iff$ $\exists N$ $L^{\otimes N}$ is very ample.

These definitions do not make sense if $X$ is an algebraic stack. But we have equivalent criteria for ampleness in case of schemes.

- Serre's vanishing of cohomology
- Global generation of coherent sheaves
- Nakai's criterion (positivity of intersection number)
- Seshadri's criterion

We generalize these criteria for algebraic stacks.

## Line Bundles on Algebraic Stacks

$\mathscr{X}$ : algebraic stack / $S = \operatorname{Spec} k$.

- A **line bundle** $L$ on $\mathscr{X}$ corresponds to a morphism

$$\mathscr{X} \to B\mathbb{G}_m = [S/\mathbb{G}_m].$$

- A **global section** of $L$ corresponds to a morphism

$$\mathscr{X} \to [\mathbb{A}^1/\mathbb{G}_m]$$

s.t. its composition with the natural projection $[\mathbb{A}^1/\mathbb{G}_m] \to B\mathbb{G}_m$ is isomorphic to $L$.

Suppose $\mathscr{X}$ is proper and has a **coarse moduli space** $M$ (Keel-Mori, Conrad). If $L$ is **basepoint free**, then generators $(s_0, s_1, \dots, s_n)$ of $\mathrm{H}^0(\mathscr{X}, L)$ corresponds to

$$\mathscr{X} \to \mathbb{P}^n \subset [\mathbb{A}^{n+1}/\mathbb{G}_m] \to B\mathbb{G}_m.$$

Therefore $L : \mathscr{X} \to B\mathbb{G}_m$ factor through $M$, that is, $L$ is a pullback of a line bundle on $M$.

## Deligne-Mumford Stacks

$\mathscr{X}$ : Deligne-Mumford stack / $\mathbb{C}$
or tame DM-stack / $k = \bar{k}$, $\mathrm{char}\, k = p > 0$.

$\mathscr{X}$ is **tame** $\iff$ for all geometric point $x$ of $\mathscr{X}$, $p \not| \# \mathrm{Aut}(x)$.

$L$ : line bundle on $\mathscr{X}$.

**Theorem**

Suppose $\mathscr{X}$ is proper, reduced and irreducible. The followings are equivalent.

1. $\exists N$, $L^{\otimes N}$ is a pullback of a **very ample** line bundle on the coarse moduli space.
2. (Serre's vanishing) For any coherent sheaf $\mathscr{F}$ on $\mathscr{X}$ and $i > 0$, there exists $N$ s.t.
$$\mathrm{H}^i(\mathscr{X}, \mathscr{F} \otimes L^{\otimes N}) = 0.$$
3. For any coherent sheaf $\mathscr{F}$ on $\mathscr{X}$, $\exists N$ s.t. $\mathscr{F} \otimes L^{\otimes N}$ is **generated by global sections**.
4. (Nakai's criterion) For any $\mathscr{Y} \subset \mathscr{X}$ : closed, reduced and irreducible substack of dimension $s$,
$$(L^s.\mathscr{Y}) > 0.$$
5. (Seshadri's criterion) $\exists \varepsilon > 0$ s.t. for any $\mathscr{C} \subset \mathscr{X}$ : closed, reduced and irreducible substack of dimension one,
$$\frac{(L.\mathscr{C})}{\sup_{p \in \mathscr{C}} \mathrm{mult}_p \mathscr{C}} > \varepsilon.$$

**Key**

The coarse moduli map $p : \mathscr{X} \to M$ has following properties.

- $p$ is proper.
- $p_*$ sends coherent sheaves to coherent sheaves.
- $p_*$ is exact. (Abramovich-Olsson-Vistoli)
- Intersection theory on $\mathscr{X}$ (Vistoli, Kresch) :

$$A_*(\mathscr{X}) \otimes \mathbb{Q} \simeq A_*(M) \otimes \mathbb{Q}.$$

**Example**

Proof of **projectivity of the coarse moduli spaces of pointed stable curves** $\overline{M_{g,n}}$ (Knudsen) : give an ample line bundle on the moduli stack $\overline{\mathcal{M}}_{g,n}$.

## Artin Stacks

For Artin stacks, these criteria do not make sense, or are not equivalent in general.

$x \in \mathscr{X}(\mathbb{C})$, $G = \mathrm{Aut}(x)$, $\mathscr{F}$ : supported on $x$. Then

$$\mathrm{H}^i(\mathscr{X}, \mathscr{F} \otimes L^{\otimes N}) = \mathrm{H}^i_{\mathrm{gr}}(G, \mathscr{F}_x).$$

Intersection multiplicities (Kreschs's intersection theory) may be zero.

But we can define a **variant of Seshadri constant** on smooth Artin stacks:

$$s(L', x) = \max\left\{ s \mid \mathrm{H}^0(\mathscr{X}, L') \twoheadrightarrow \mathrm{H}^0(\mathscr{X}, L' \otimes \mathcal{O}_{\mathscr{X}}/m_x{}^{s+1}) \right\},$$

$$\sigma(L, x) = \limsup_{k \to \infty} \frac{s(L^{\otimes k}, x)}{k}.$$

For a basepoint free line bundle $L$ on proper reduced algebraic stack $\mathscr{X}$ over $\mathbb{C}$, we set $Z_L$ to be the image of $\varphi_L : \mathscr{X} \to \mathbb{P}^n$.

**Theorem**

If $L$ satisfies $\inf_x \sigma(L, x) > 0$ ("ample"), then for $N \gg 0$, the projective scheme $Z_{L^{\otimes N}}$ does not depend on $L$, i.e. if $L$ and $L'$ are "ample" line bundles, $\exists N, N'$ s.t. $Z_{L^{\otimes N}} \simeq Z_{L'^{\otimes N'}}$.

**proof**

We have a diagram

$$\begin{tikzcd}
\mathscr{X} \arrow[r] \arrow[dr] & Z_{L^{\otimes N} \otimes L'^{\otimes N'}} \arrow[r, hook] \arrow[d] & \mathbb{P}^{n''} \arrow[drr] & & \\
& Z_{L^{\otimes N}} \times Z_{L'^{\otimes N'}} \arrow[r, hook] & \mathbb{P}^n \times \mathbb{P}^{n'} \arrow[rr, hook] & & \mathbb{P}^{(n+1)(n'+1)-1}.
\end{tikzcd}$$

If $\inf \sigma > 0$ and $N, N' \gg 0$, the composition

$$Z_{L^{\otimes N} \otimes L'^{\otimes N'}} \to Z_{L^{\otimes N}} \times Z_{L'^{\otimes N'}} \to Z_{L^{\otimes N}}$$

is surjective and separates points and tangents, hence isomorphic.

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